

# Uniform Asymptotics in the Problem of Superfluidity of Classical Liquids in Nanotubes

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## Abstract

In the preceding papers (see [1, 2]), the superfluidity of the classical liquid was proved under the assumption that the parameters  $N$  and  $r$ , where  $N$  is the particle number and  $r$  is the capillary radius, tend respectively to infinity and to zero so that  $\frac{1}{N} \ll \frac{r}{R}$ , where  $R$  is the capillary length. In the present paper, this assumption is removed.

1. We first note that solutions of the variational equation for the Vlasov equation do not coincide with the classical limit for the variational equations corresponding to the mean-field equations in quantum theory. We consider mean-field equations of the form

$$ih \frac{\partial}{\partial t} \varphi^t(x) = \left( -\frac{\hbar^2}{2m} \Delta + W_t(x) \right) \varphi^t(x), \quad W_t(x) = U(x) + \int V(x, y) |\varphi^t(y)|^2 dy, \quad (1)$$

with the initial condition  $\varphi|_{t=0} = \varphi_0$ , where  $\varphi_0 \in W_2^\infty(\mathbf{R}^\nu)$ ,  $\int dx |\varphi_0(x)|^2 = 1$ .

To find an asymptotic representation of the complex-germ type [3], we consider a system consisting of the Hartree equation and its conjugate equation. We then take the system of variational equations for it and replace the variations  $\delta\varphi$  and  $\delta\varphi^*$  with the independent functions  $F$  and  $G$ . For  $F$  and  $G$ , we obtain the system of equations:

$$\begin{aligned} i \frac{\partial F^t(x)}{\partial t} &= \int dy \left( \frac{\delta^2 H}{\delta \varphi^*(x) \delta \varphi(y)} F^t(y) + \frac{\delta^2 H}{\delta \varphi^*(x) \delta \varphi^*(y)} G^t(y) \right); \\ -i \frac{\partial G^t(x)}{\partial t} &= \int dy \left( \frac{\delta^2 H}{\delta \varphi(x) \delta \varphi(y)} F^t(y) + \frac{\delta^2 H}{\delta \varphi(x) \delta \varphi^*(y)} G^t(y) \right). \end{aligned} \quad (2)$$

Roughly speaking, the classical equations can be obtained from the quantum ones using a substitution of the form  $\varphi = \chi e^{\frac{i}{\hbar} S}$  (the WKB method),  $\varphi^* = \chi^* e^{\frac{i}{\hbar} S^*}$ ,  $S = S^*$ ,  $\chi = \chi(x, t) \in C^\infty$ ,  $S = S(x, t) \in C^\infty$ .

For variational equations, it is natural to vary not only the limit equation for  $\chi$  and  $\chi^*$  but also the functions  $S$  and  $S^*$ . This gives an important new term in the solution of the equation for collective oscillations. We consider this fact for the simplest example investigated in the famous work by Bogoliubov on a weakly nonideal Bose gas [4].

Let  $U = 0$  for (1) in a three-dimensional box with edge length  $L$ , and let the  $L$ -periodicity condition be imposed on the wave functions in this case (i.e., a problem on the torus with generators  $L, L$ , and  $L$  is considered). Then the function

$$\varphi(x) = L^{-3/2} e^{i/h(px - \Omega t)}, \quad (3)$$

with  $p = 2\pi n/L$ , where  $n$  is an integer-valued vector, satisfies (1) with

$$\Omega = \frac{p^2}{2m} + L^{-3} \int dx V(x). \quad (4)$$

We consider functions  $F^{(\lambda)}(x)$  and  $G^{(\lambda)}(x)$ , where  $\lambda = 2\pi n/L$ ,  $n \neq 0$  of the form

$$\begin{aligned} F^{(\lambda)t}(x) &= L^{-3/2} \rho_\lambda e^{\frac{i}{\hbar}[(p+\lambda)x + (\beta-\Omega)t]}, \\ G^{(\lambda)t}(x) &= L^{-3/2} \sigma_\lambda e^{\frac{i}{\hbar}[(-p+\lambda)x + (\beta+\Omega)t]}, \end{aligned} \quad (5)$$

where

$$\begin{aligned} -\beta_\lambda \rho_\lambda &= \left( \frac{(p+\lambda)^2}{2m} - \frac{p^2}{2m} + \tilde{V}_\lambda \right) \rho_\lambda + \tilde{V}_\lambda \sigma_\lambda, \\ \beta_\lambda \rho_\lambda &= \left( \frac{(p-\lambda)^2}{2m} - \frac{p^2}{2m} + \tilde{V}_\lambda \right) \sigma_\lambda + \tilde{V}_\lambda \rho_\lambda, \\ |\sigma_\lambda|^2 - |\rho_\lambda|^2 &= 1, \quad \tilde{V}_\lambda = L^{-3} \int dx V(x) e^{\frac{i}{\hbar}\lambda x}. \end{aligned} \quad (6)$$

From system (6) we obtain

$$\beta_\lambda = -p\lambda + \sqrt{\left( \frac{\lambda^2}{2m} + \tilde{V}_\lambda \right)^2 - \tilde{V}_\lambda^2}. \quad (7)$$

In this example, we have  $u = e^{\frac{i}{\hbar}s(x,t)}$  and  $u^* = e^{-\frac{s(x,t)}{\hbar}}$ , where  $s(x,t) = px + \beta t$ , and the variation of the action for the vector  $(\delta u, \delta u^*)$  is equal to  $\lambda x \pm \Omega t$ .

A more thorough passage to the limit gives  $\tilde{V}_\lambda \rightarrow V_0 = L^{-3} \int dx V(x)$ .

In the classical limit, we thus obtain the famous Bogoliubov relation (7). In this case, we have  $u(x) = 0$ , and the exact solution coincides with the classical one as in the linear Schrödinger equation. The situation with  $u(x) \neq 0$  was investigated in [5], and it turns out that a relation similar to (7) is the classical limit as  $\hbar \rightarrow 0$  for the variational equation in this general case. The curve for the dependence of  $\beta_\lambda$  on  $\lambda$  is called the Landau curve, and it specifies the superfluid state. The value  $\lambda_{cr}$  at which the superfluidity disappears is called the Landau criterion. Bogoliubov explains the superfluidity phenomenon as follows: “the ‘degenerate condensate’ can move without friction relative to elementary excitations with an arbitrary sufficiently small velocity” ([4], p.210).

But this mathematical consideration is not related to the Bose-Einstein condensate; merely the quasi-particle spectrum determined for  $\lambda < \lambda_{cr}$  is positive. This means that it is metastable (see [6]). The Bose-Einstein condensate is mentioned here only to disprove the idea that it follows from what was said above that this consideration applies to a classical liquid.

Indeed, for example, the molecules of a classical undischarged liquid are Bose particles if the number of neutrons in the molecule is even. Because every particle (molecule) is neutral and is formed of an even number  $l$  of neutrons, an  $N$ -particle equation can be written for this liquid. Thus, every its particle is  $3(2k+l)$ -dimensional, where  $k$  is the number of electrons; there is a dependence on the potential  $u(x_i)$ ,  $x_i \in R^{6k+3l}$ ; and an equation for  $N$  particles  $x_i, i = 1, \dots, N$  with a pair interaction potential  $V(x_i - x_j)$  can be considered.

But Bogoliubov found only one series for the spectrum of the many-particle problem. As Landau wrote, “N. N. Bogoliubov recently managed to find the general form of the

energy spectrum for the Bose–Einstein gas with a weak interaction between the particles using a clever application of second quantization” ([7], p. 43). But this series is not unique, i.e., it does not exhaust the whole energy spectrum.

In 2001, we suggested the ultrasecond quantization method [8] (also see [9, 10, 11, 12, 13]). The ultrasecond-quantized Schrödinger equations, like the second-quantized ones, represent the  $N$ -particle Schrödinger equation, and this means that the ultrasecond-quantized equation is essentially identical to the original  $N$ -particle equation: it coincides with the latter on the  $3N$ -dimensional space. But in contrast to the second-quantized case, replacing the creation and annihilation operators with  $c$ -numbers does not yet give the correct asymptotic representation; it turns out that its results coincide with those obtained by applying the Schröder variational principle or the Bogoliubov variational method.

For the Bardeen exotic potential, the correct asymptotic solution coincides with the one resulting from applying the abovementioned ultrasecond quantization method. For potentials of general form, in the case of pair interaction for example, the answer turns out different. In particular, the ultrasecond quantization method gives some other asymptotic series of eigenvalues corresponding to the  $N$ -particle Schrödinger equation, which, in contrast to Bogoliubov series (7), are not metastable. They correspond to vortex filaments [14].

It turns out that the decisive factor here is not the Bose-Einstein condensate but the thickness of the capillary (nanotube) in which the liquid flows. If we consider a liquid in a capillary or nanotube, then the velocity corresponding to metastable states is not small for a sufficiently small radius. Consequently, the liquid flows without friction for a smaller velocity.

The no-flow condition on the boundary of the nanotube (absence of flow) is the Dirichlet boundary condition or the Born–von Karman boundary condition. It generates a standing wave that can be interpreted as a particle-antiparticle pair: a particle with the momentum  $p$  orthogonal to the tube wall and an antiparticle with the momentum  $-p$ .

In the boson case, we consider a short-range interaction potential  $V(x_i - x_j)$ . This means that only interaction with finitely many particles is possible as  $N \rightarrow \infty$  ( $N$  is the number of particles). Consequently, the potential depends on  $N$  as  $V_N = V((x_i - x_j)N^{1/3})$ . If  $V(y)$  is finitely supported in  $\Omega_V$ , then the number of particles captured by the support is independent of  $N$  as  $N \rightarrow \infty$ . As result, superfluidity occurs for velocities less than  $\min(\lambda_{cr}, \frac{\hbar}{2mR})$ , where  $R$  is the nanotube radius. The upper bound is determined by the condition that the radius of action of the molecule must be less than the radius of the nanotube.

We now present our own considerations that do not relate to the mathematical presentation. Viscosity is connected with collisions of particles: the higher the temperature is, the greater the number of collisions. In a nanotube, there are few collisions because only those with the tube walls occur, which is taken into account by the series obtained below. Precisely this fact rather than the presence of the Bose condensate leads to the weakening of viscosity and consequently to superfluidity. In other words, even for liquid He4, the main factor in the superfluidity phenomenon is not the condensate but the presence of a thin capillary [15, 16].

**2.** In this part of the paper we refine solutions of equations in variations (26)-(28) and (35)-(37) presented in the paper [18]; see also [17] and [19].

For the Fermi liquid (for instance, for helium 3), we solve the mathematical problem on the reduction of the  $N$ -particle Schrödinger equation with the pair interaction potential typical for helium: repulsion as a pair of particles approaches and attraction as the

pair of particles moves away. This means that we do not take into account the possible resonance interaction of Cooper pairs, as is usual in superconductivity problems. Moreover, our assumption forbids the radius of the capillary to be less than the radius of the molecule itself (otherwise the molecule simply cannot enter the capillary). However, as the radius decreases, a superfluidity domain can occur for a liquid with an even number of neutrons. Nevertheless, for a Fermi liquid, superfluidity also occurs, but in quasithermodynamics rather than in the thermodynamical limit. The notion of quasithermodynamics was recently introduced by the author [20].

**3.** The eigenvalues of the equations in variations for Bose-liquid are [20]:

$$\lambda_{1,k_1k_2,l} = -\frac{\hbar^2}{m}k_1(k_2 + l) + \sqrt{\frac{\xi_{k_2,l} + \sqrt{\xi_{k_2,l}^2 - 4\eta_{k_2,l}}}{2}}, \quad (8)$$

where

$$\begin{aligned} \xi_{k_2,l} = & a^2 \left( (l_1^2 - k_2^2)^2 + (l^2 - k_2^2)^2 \right) + \\ & + a \left( l_1^2 (v_{l+3k_2} - v_{2k_2}) + l^2 (v_{l-k_2} - v_{2k_2}) - k_2^2 (v_{l-k_2} + v_{l+3k_2} - 2v_{2k_2}) \right) - \\ & - (v_{l+k_2} + v_{2k_2})(v_{l+3k_2} + v_{l-k_2} - 2v_{2k_2})/2, \\ \eta_{k_2,l} = & a \left( 2a (k_2^4 - k_2^2 (l_1^2 + l^2) + l_1^2 l^2) - (l_1^2 + l^2 - 2k_2^2) (v_{l+k_2} + v_{2k_2}) \right) \cdot \\ & \cdot \left( 2a^2 (k_2^4 - k_2^2 (l_1^2 + l^2) + l_1^2 l^2) + \right. \\ & + a (l_1^2 (2v_{l-k_2} + v_{l+k_2} - v_{2k_2}) + l^2 (2v_{l+3k_2} + v_{l+k_2} - v_{2k_2}) - \\ & - 2k_2^2 (v_{l+3k_2} + v_{l-k_2} + v_{l+k_2} - v_{2k_2})) + \\ & \left. + 2(v_{l+3k_2} - v_{2k_2})(v_{l-k_2} - v_{2k_2}) + (v_{l+k_2} + v_{2k_2})(v_{l+3k_2} + v_{l-k_2} - 2v_{2k_2}) \right) / 4; \end{aligned}$$

and where the notation

$$a = \frac{\hbar^2}{2m}, \quad l_1 = l + 2k_2.$$

was used for brevity.

Relation (8), as compared with (27) in [17], is uniform with respect to  $k_2$  as  $k_2 \rightarrow \infty$ . Note that, if  $k_2 = 0$ , then the Bogolyubov relation holds,

$$\lambda_{1,k_1,l} = -\frac{\hbar^2}{m}k_1l + \sqrt{\left(\frac{\hbar^2 l^2}{2m} + v_l\right)^2 - v_l^2}. \quad (9)$$

For a system of identical Fermi-particles, the eigenvalue problem for the system of equations in variations can similarly be reduced to the problem of finding the eigenvalues of the equation

$$\tilde{\lambda}X = MX. \quad (10)$$

Here  $\tilde{\lambda} = \lambda + \frac{\hbar^2}{m}k_1(k_2 + l)$ ,  $X$  - is a column vector of the form

$$X = \begin{pmatrix} u_{1,l} \\ u_{2,l} \\ v_{1,l} \\ v_{2,l} \end{pmatrix},$$

$M$  – stands for the matrix

$$M = \begin{pmatrix} B_1 & V & V_1 & 0 \\ V & B_2 & 0 & V_2 \\ M_1 & F & -B_1 & -V \\ F & M_2 & -V & -B_2 \end{pmatrix}$$

with the elements

$$\begin{aligned} B_1 &= B_{k_2, l} + \frac{v_{l-k_2}}{2}, & V &= \frac{v_{l+k_2} - v_{2k_2}}{2}, \\ B_2 &= B_{k_2, l+2k_2} + \frac{v_{l+3k_2}}{2}, & V_1 &= \frac{v_{l-k_2} - v_{l+k_2}}{2}, \\ M_1 &= 2i(v_{l-k_2} - v_0)\varphi_{k_2, l}, & V_2 &= \frac{v_{l+3k_2} - v_{l+k_2}}{2}, \\ M_2 &= 2i(v_0 - v_{l+3k_2})\varphi_{k_2, l+2k_2}, & F &= i(v_{2k_2} - v_{l+k_2})(\varphi_{k_2, l+2k_2} - \varphi_{k_2, l}), \end{aligned} \quad (11)$$

where the numbers  $B_{k_2, l}$  and  $\varphi_{k_2, l}$  are

$$\begin{aligned} B_{k_2, l} &= \frac{\hbar^2}{2m}(l^2 - k_2^2) + i(v_{l+k_2} - v_{l-k_2})\varphi_{k_2, l} - \frac{v_{2k_2}}{2}, \\ \varphi_{k_2, l} &= -\frac{ib_{k_2, l}}{2} \pm \frac{i\sigma_l}{2}\sqrt{b_{k_2, l}^2 - 1}, & b_{k_2, l} &\equiv \frac{\frac{\hbar^2}{m}(l^2 - k_2^2) + (v_0 - v_{2k_2})}{v_{l-k_2} - v_{l+k_2}}, \end{aligned} \quad (12)$$

$k_1$ ,  $k_2$  and  $l$  are three-dimensional vectors of the form

$$2\pi \left( \frac{n_1}{L_1}, \frac{n_2}{L_2}, \frac{n_3}{L_2} \right), \quad (13)$$

$n_1$ ,  $n_2$  and  $n_3 - n_3$  are integers. The summation is taken over all values of  $n_1$ ,  $n_2$ ,  $n_3$ , and

$$\begin{aligned} \tilde{\xi}_{k_2, l} &= a^2 \left( (l_1^2 - k_2^2)^2 + (l^2 - k_2^2)^2 \right) + \\ &+ a \left( l_1^2 (v_{l+3k_2} - v_{2k_2}) + l^2 (v_{l-k_2} - v_{2k_2}) - k_2^2 (v_{l-k_2} + v_{l+3k_2} - 2v_{2k_2}) \right) + \\ &+ (v_{l+k_2} - v_{2k_2})(v_{l-k_2} + v_{l+3k_2} - 2v_{2k_2})/2, \\ \tilde{\eta}_{k_2, l} &= a \left( 2a (k_2^4 - k_2^2 (l_1^2 + l^2) + l_1^2 l^2) + (l_1^2 + l^2 - 2k_2^2) (v_{l+k_2} - v_{2k_2}) \right) \cdot \\ &\cdot \left( 2a^2 (k_2^4 - k_2^2 (l_1^2 + l^2) + l_1^2 l^2) + \right. \\ &+ a (l_1^2 (2v_{l-k_2} - v_{2k_2} - v_{l+k_2}) + l^2 (2v_{l+3k_2} - v_{2k_2} - v_{l+k_2}) - \\ &- 2k_2^2 (v_{l+3k_2} + v_{l-k_2} - v_{l+k_2} - v_{2k_2})) + \\ &\left. + 2(v_{l+3k_2} - v_{2k_2})(v_{l-k_2} - v_{2k_2}) - (v_{l+k_2} - v_{2k_2})(v_{l+3k_2} + v_{l-k_2} - 2v_{2k_2}) \right) / 4, \end{aligned}$$

where the notation

$$a = \frac{\hbar^2}{2m}, \quad l_1 = l + 2k_2.$$

is used for brevity. The eigenvalues of the system of equations in variations are

$$\lambda_{1, k_1 k_2, l} = -\frac{\hbar^2}{m} k_1 (k_2 + l) + \sqrt{\frac{\tilde{\xi}_{k_2, l} + \sqrt{\tilde{\xi}_{k_2, l}^2 - 4\tilde{\eta}_{k_2, l}}}{2}}. \quad (14)$$

For  $k_2 = 0$ , i.e., along a capillary, the superfluidity in the quasithermodynamics holds and, for  $k_2 \neq 0$ , i.e., if there is a reflection from the walls of the capillary, then an instability occurs because the eigenvalues become complex by (14). This gives the same upper bound for the critical speed as that in the Bose-case (due to occurrence of a vortex).

According to [17, 18, 19] only the thermodynamical and quasithermo-dynamical limits in statistical physics exist. The above solution on the superfluidity of a Fermi liquid enables us to claim that the phase transition in this problem from the superfluid state to the normal one is an example of a phase transition in quasithermodynamics.

The results obtained in this paper are based on the author's paper of the year 1995 [5] which is given in appendix below.

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# Quasi-Particles Associated with Lagrangian Manifolds Corresponding to Semiclassical Self-Consistent Fields. III <sup>1</sup>

In the preceding part of this paper, we presented Eqs. (25) for quasi-particles associated with an  $n$ -dimensional Lagrangian manifold and Eq. (29) for quasi-particles corresponding to a  $(2n - 1)$ -dimensional manifold. These equations were written out only in the  $x$ -chart, and the quantum corrections were given without proof. In this part we essentially use the canonical operator method to obtain Eq. (25) with corrections in the  $x$ -chart as well as in any other chart of the canonical atlas [1]. To derive the correction in Eq. (29), a “modified”  $\delta$ -function must be used, and this will be done in the next part of the paper.

To obtain the result in an arbitrary canonical chart, one should pass on to the  $p$ -representation with respect to some of the coordinates in the Hartree equation. This is actually equivalent [2] to considering the Hartree-type equation

$$\left[ H_0 \left( \overset{2}{x}, -ih \overset{1}{\frac{\partial}{\partial x}} \right) + \int dy \psi^*(y) H_1 \left( \overset{2}{x}, -ih \overset{1}{\frac{\partial}{\partial x}}; \overset{2}{y}, -ih \overset{1}{\frac{\partial}{\partial y}} \right) \psi(y) \right] \psi(x) = \Omega \psi(x), \quad (\text{A.1})$$

where  $x, y \in \mathbb{R}^n$ ,  $\psi \in L^2(\mathbb{R}^n)$  is a complex-valued function,  $h > 0$ ,  $\Omega \in \mathbb{R}$ , and the indices 1 and 2 specify the ordering of the operators  $x$  and  $-ih\partial/\partial x$ . The function  $H_1$  satisfies the condition  $H_1(x, p_x; y, p_y) = H_1(y, p_y; x, p_x)$ . Equation (A.1) generalizes the ordinary Hartree equation (Eq.(1) in [4], where  $N = 1$ ). The study of Eq. (A.1) is important, for example, if one makes an attempt to find a solution to the Hartree equation (1) in the momentum representation,

$$\psi(x) = \int \tilde{\psi}(p) e^{(i/\hbar)px} \frac{dp}{(2\pi\hbar)^{n/2}}.$$

Let us also discuss the variational system associated with Eq. (A.1), which can be obtained as follows. Along with Eq. (A.1), let us write out the conjugate equation and consider the variations of both equations *assuming that the variations  $\delta\psi = F$  and  $\delta\psi^* = G$  are independent*.

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<sup>1</sup>V.P.Maslov, Russian J. Math. Phys. 1995, v.3, N3, 401-406.



The variational system has the form

$$\begin{aligned}
& \left[ H_0 \left( \overset{2}{x}, -ih \overset{1}{\frac{\partial}{\partial x}} \right) - \Omega + \int dy \psi^*(y) H_1 \left( \overset{2}{x}, -ih \overset{1}{\frac{\partial}{\partial x}}; \overset{2}{y}, -ih \overset{1}{\frac{\partial}{\partial y}} \right) \psi(y) \right] F(x) \\
& + \int dy \left( G(y) H_1 \left( \overset{2}{x}, -ih \overset{1}{\frac{\partial}{\partial x}}; \overset{2}{y}, -ih \overset{1}{\frac{\partial}{\partial y}} \right) \psi(y) \right. \\
& \left. + \psi^*(y) H_1 \left( \overset{2}{x}, -ih \overset{1}{\frac{\partial}{\partial x}}; \overset{2}{y}, -ih \overset{1}{\frac{\partial}{\partial y}} \right) F(y) \right) \psi(x) = -\beta F(x), \tag{A.2} \\
& \left[ H_0 \left( \overset{1}{x}, ih \overset{2}{\frac{\partial}{\partial x}} \right) - \Omega + \int dy \psi(y) H_1 \left( \overset{1}{x}, ih \overset{2}{\frac{\partial}{\partial x}}; \overset{1}{y}, ih \overset{2}{\frac{\partial}{\partial y}} \right) \psi^*(y) \right] G(x) \\
& + \int dy \left( F(y) H_1 \left( \overset{1}{x}, ih \overset{2}{\frac{\partial}{\partial x}}; \overset{1}{y}, ih \overset{2}{\frac{\partial}{\partial y}} \right) \psi^*(y) \right. \\
& \left. + \psi(y) H_1 \left( \overset{1}{x}, ih \overset{2}{\frac{\partial}{\partial x}}; \overset{1}{y}, ih \overset{2}{\frac{\partial}{\partial y}} \right) G(y) \right) \psi^*(x) = \beta G(x).
\end{aligned}$$

Equations (A.1) and (A.2) play an important role in the problem of constructing asymptotic solutions to the  $N$ -particle Schrödinger equation as  $N \rightarrow \infty$  [5]–[7].

For example, the spectrum of system (A.2) (possible values of  $\beta$ ) corresponds to the spectrum of quasi-particles. Namely, the difference between the energy of an excited state and the ground state energy is given by the expression  $\sum_k \beta_k n_k$ , where the numbers  $n_k \in \mathbb{Z}_+$ ,  $k = \overline{1, \infty}$ , which are equal to zero starting from some  $k$ , define the eigenfunction and the eigenvalue of the excited state, and  $\beta_k \in \mathbb{R}$  are the eigenvalues of system (A.2).

In this paper we are interested in asymptotic solutions to Eqs. (A.1) and (A.2) as the “inner”  $h$  tends to zero.

Asymptotic solutions to Eq. (A.1) are given [8] by the canonical operator on a Lagrangian manifold  $\Lambda^n = \{x = X(\alpha), p = P(\alpha)\}$  invariant with respect to the Hamiltonian system

$$\dot{x} = \frac{\partial H(x, p)}{\partial p}, \quad \dot{p} = -\frac{\partial H(x, p)}{\partial x}, \tag{A.3}$$

where

$$H(x, p) = H_0(x, p) + \int d\mu_\alpha H_1(x, p; X(\alpha), P(\alpha)),$$

$\alpha \in \Lambda^n$ , and  $d\mu_\alpha$  is an invariant measure on  $\Lambda^n$ . The Lagrangian manifold lies on the surface  $H(x, p) = \Omega$ . If a chart  $A$  is projected diffeomorphically in the  $x$ -plane, then the canonical operator acts as the multiplication by  $\exp\{(i/h)S(x)\}/\sqrt{J}$ , where  $S(x) = \int p dx$  on  $\Lambda^n$  and  $J = Dx/D\mu_\alpha$ . We are interested in finding asymptotic solutions to Eqs. (A.2). Without loss of generality, we can confine ourselves to the case of  $x$ -chart. Indeed, to obtain similar expressions in the  $p$ -chart, one must consider the Fourier transformation of Eqs. (A.1) and (A.2) and apply the same technique, since the form of the equations remains unchanged.

Let us seek the asymptotic solutions to Eqs. (A.2) in the  $x$ -chart in the form

$$F(x) = \tilde{f}(x)\psi(x), \quad G(x) = \tilde{g}(x)\psi^*(x), \tag{A.4}$$

where the functions  $f$  and  $g$ , in contrast to  $\psi$  and  $\psi^*$ , have a limit as  $h \rightarrow 0$ . One can consider a more general case, by allowing  $f$  and  $g$  to be functions of  $x$  and  $-ih\partial/\partial x$ , but

in the leading term as  $h \rightarrow 0$  we have

$$-ih \frac{\partial}{\partial x} e^{(i/h)S} \approx \frac{\partial S}{\partial x} e^{(i/h)S},$$

and so we arrive at functions  $f$  and  $g$  that depend only on  $x$ .

The second equation in system (A.2) can be rewritten in the form

$$\begin{aligned} & \left[ H_0 \left( x, ih \frac{\partial}{\partial x} \right) + \int dy \psi(y) H_1 \left( x, ih \frac{\partial}{\partial x}; y, ih \frac{\partial}{\partial y} \right) \psi^*(y); \tilde{g}(x) \right] \psi^*(x) \\ & + \int dy \left\{ \psi(y) \tilde{c}(y) H_1 \left( x, ih \frac{\partial}{\partial x}; y, ih \frac{\partial}{\partial y} \right) \psi^*(y) \right. \\ & \left. + \psi(y) \left[ H_1 \left( x, ih \frac{\partial}{\partial x}; y, ih \frac{\partial}{\partial y} \right); \tilde{g}(y) \right] \psi^*(y) \right\} \psi^*(x) = \beta \tilde{g}(x) \psi^*(x), \end{aligned} \quad (\text{A.5})$$

where  $[A; B] = AB - BA$  and

$$c(x) = \tilde{f}(x) + \tilde{g}(x). \quad (\text{A.6})$$

Equation (A.1) is used in the derivation of Eq. (A.5). We observe that all terms containing the function  $\tilde{g}$  on the left-hand side in Eq. (A.1) are  $O(h)$ , since the commutator of two operators depending on  $x$  and  $-ih\partial/\partial x$  is equal, in the classical limit, to  $(-ih)$  times the Poisson bracket of the corresponding classical quantities.

Thus, the function  $c$ , as well as the eigenvalue  $\beta$ , is assumed to be  $O(h)$ . Let us rescale these quantities as follows:

$$c(x) = h\tilde{c}(x), \quad \beta = h\tilde{\beta}. \quad (\text{A.7})$$

Now we can derive the equation for  $\tilde{g}$ ,  $\tilde{c}$ , and  $\tilde{\beta}$  in the leading term in  $h$  and the first correction to it from Eq. (A.5), making use of the following relations:

$$\text{i) } \left[ A \left( x, ih \frac{\partial}{\partial x} \right); \xi(x) \right] = \sum_{a=1}^n ih \frac{\partial A}{\partial p_a} \left( x, ih \frac{\partial}{\partial x} \right) \frac{\partial \xi}{\partial x_a} - \sum_{a,b=1}^n \frac{h^2}{2} \frac{\partial^2 A}{\partial p_a \partial p_b} \left( x, ih \frac{\partial}{\partial x} \right) \frac{\partial^2 \xi}{\partial x_a \partial x_b}, \quad (\text{A.8})$$

where  $p_a = ih\partial/\partial x_a$ ,  $A(x, p)$  is a function  $\mathbb{R}^{2n} \rightarrow C$ ,  $\xi : \mathbb{R}^n \rightarrow \mathbb{R}$ ;

ii)  $\psi(x) = \chi(x, h) e^{(i/h)S(x)}$ , where  $\chi = 1/\sqrt{J}$  in the leading term in  $h$ ;

$$\text{iii) } ih \frac{\partial}{\partial x} e^{-(i/h)S(x)} = e^{-(i/h)S(x)} \left( \frac{\partial S}{\partial x} + ih \frac{\partial}{\partial x} \right);$$

$$\begin{aligned} \text{iv) } B \left( ih \frac{\partial}{\partial x} + \frac{\partial S}{\partial x} \right) &= B \left( \frac{\partial S}{\partial x} \right) + ih \sum_{a=1}^n \frac{\partial B}{\partial p_a} \frac{\partial}{\partial x_a} + \frac{ih}{2} \sum_{a,b=1}^n \frac{\partial^2 B}{\partial p_a \partial p_b} \frac{\partial^2 S}{\partial x_a \partial x_b} \\ &+ \frac{(ih)^2}{2} \sum_{a,b=1}^n \frac{\partial^2 B}{\partial p_a \partial p_b} \frac{\partial^2}{\partial x_a \partial x_b} + \frac{(ih)^2}{2} \sum_{a,b,c=1}^n \frac{\partial^3 B}{\partial p_a \partial p_b \partial p_c} \frac{\partial^2 S}{\partial x_a \partial x_b} \frac{\partial}{\partial x_c} \\ &+ \frac{(ih)^2}{6} \sum_{a,b,c=1}^n \frac{\partial^3 B}{\partial p_a \partial p_b \partial p_c} \frac{\partial^3 S}{\partial x_a \partial x_b \partial x_c} \\ &+ \frac{(ih)^2}{8} \sum_{a,b,c,d=1}^n \frac{\partial^4 B}{\partial p_a \partial p_b \partial p_c \partial p_d} \frac{\partial^2 S}{\partial x_a \partial x_b} \frac{\partial^2 S}{\partial x_c \partial x_d} + O(h^3), \end{aligned} \quad (\text{A.9})$$

where all derivatives of  $B$  are evaluated at the point  $p = \partial S/\partial x$ .

These relations can easily be obtained for monomial functions  $A$  and  $B$ . An application of formulas i)–iv) yields the equation

$$\begin{aligned}
& i \sum_{a=1}^n \frac{\partial H}{\partial p_a^x} \frac{\partial \tilde{g}}{\partial x_a}(X(\alpha)) - \tilde{\beta} \tilde{g}(X(\alpha)) + \int d\mu_\beta \tilde{c}(X(\beta)) H_1 \\
& + i \int d\mu_\beta \sum_{a=1}^n \frac{\partial \tilde{g}}{\partial x_a}(X(\beta)) \frac{\partial H_1}{\partial p_a^y} + \frac{h}{2} \sum_{a,b=1}^n \frac{\partial \tilde{g}}{\partial x_a}(X(\alpha)) \frac{\partial^2 H}{\partial p_a^x \partial p_b^x} \frac{\partial \ln J}{\partial x_b}(X(\alpha)) \\
& - \frac{h}{2} \sum_{a,b,c=1}^n \frac{\partial \tilde{g}}{\partial x_a}(X(\alpha)) \frac{\partial^3 H}{\partial p_a^x \partial p_b^x \partial p_c^x} \frac{\partial^2 S}{\partial x_b \partial x_c}(X(\alpha)) - \frac{h}{2} \sum_{a,b=1}^n \frac{\partial^2 \tilde{g}}{\partial x_a \partial x_b}(X(\alpha)) \frac{\partial^2 H}{\partial p_a^x \partial p_b^x} \\
& + \frac{ih}{2} \int d\mu_\beta \tilde{c}(X(\beta)) \sum_{a,b=1}^n \left[ \frac{\partial^2 H_1}{\partial p_a^x \partial p_b^x} \frac{\partial^2 S}{\partial x_a \partial x_b}(X(\alpha)) + \frac{\partial^2 H_1}{\partial p_a^y \partial p_b^y} \frac{\partial^2 S}{\partial x_a \partial x_b}(X(\beta)) \right] \\
& - \frac{ih}{2} \int d\mu_\beta \tilde{c}(X(\beta)) \sum_{a=1}^n \left[ \frac{\partial H_1}{\partial p_a^x} \frac{\partial \ln J}{\partial x_a}(X(\alpha)) + \frac{\partial H_1}{\partial p_a^y} \frac{\partial \ln J}{\partial x_a}(X(\beta)) \right] \\
& + \frac{h}{2} \int d\mu_\beta \sum_{a=1}^n \frac{\partial \tilde{g}}{\partial x_a}(X(\beta)) \left\{ \sum_{b=1}^n \left( \frac{\partial^2 H_1}{\partial p_a^y \partial p_b^y} \frac{\partial \ln J}{\partial x_b}(X(\beta)) + \frac{\partial^2 H_1}{\partial p_a^y \partial p_b^x} \frac{\partial \ln J}{\partial x_b}(X(\alpha)) \right) \right. \\
& \left. - \sum_{b,c=1}^n \left( \frac{\partial^3 H_1}{\partial p_a^y \partial p_b^x \partial p_c^x} \frac{\partial^2 S}{\partial x_b \partial x_c}(X(\alpha)) + \frac{\partial^3 H_1}{\partial p_a^y \partial p_b^y \partial p_c^y} \frac{\partial^2 S}{\partial y_b \partial y_c}(X(\beta)) \right) \right\} \\
& - \frac{h}{2} \int d\mu_\beta \sum_{a,b=1}^n \frac{\partial^2 H_1}{\partial p_a^y \partial p_b^y} \frac{\partial^2 g}{\partial x_a \partial x_b}(X(\beta)) = 0; \tag{A.10}
\end{aligned}$$

in this formula the arguments

$$x = X(\alpha), \quad p^x = P(\alpha), \quad y = X(\beta), \quad p^y = P(\beta) \tag{A.11}$$

of the function  $H_1$  and of its derivatives, as well as the arguments  $x = X(\alpha)$ ,  $p^x = P(\alpha)$  of the function  $H$ , are omitted.

Let us now find another equation relating  $\tilde{g}$  to  $\tilde{c}$ . To this end, let us multiply the first equation in system (A.2) by  $\psi^*(x)$  and the second equation by  $\psi(x)$ . Let us subtract the first product from the second. We obtain

$$\begin{aligned}
& \beta \psi^*(x) \psi(x) c(x) = \psi(x) \left[ H\left(x, ih \frac{\partial}{\partial x}\right); \tilde{g}(x) \right] \psi^*(x) \\
& + \psi^*(x) \left[ H\left(x, -ih \frac{\partial}{\partial x}\right); \tilde{g}(x) \right] \psi(x) - \psi^*(x) \left[ H\left(x, -ih \frac{\partial}{\partial x}\right); \tilde{c}(x) \right] \psi(x) \\
& + \psi(x) \int dy \psi(y) \left[ H_1\left(x, ih \frac{\partial}{\partial x}; y, ih \frac{\partial}{\partial y}\right); \tilde{g}(y) \right] \psi^*(y) \psi^*(x) \\
& - \psi^*(x) \int dy \psi^*(y) \left[ \tilde{g}(y); H_1\left(x, -ih \frac{\partial}{\partial x}; y, -ih \frac{\partial}{\partial y}\right) \right] \psi(y) \psi(x) \\
& + \psi(x) \int dy \psi(y) c(y) H_1\left(x, ih \frac{\partial}{\partial x}; y, ih \frac{\partial}{\partial y}\right) \psi^*(y) \psi^*(x) \\
& - \psi^*(x) \int dy \psi^*(y) H_1\left(x, -ih \frac{\partial}{\partial x}; y, -ih \frac{\partial}{\partial y}\right) c(y) \psi(y) \psi(x). \tag{A.12}
\end{aligned}$$

Let us use Eqs. (A.8)–(A.10). We find the following equation for  $\tilde{g}$  and  $\tilde{c}$  modulo  $O(h^2)$ :

$$\begin{aligned}
& i \sum_{a=1}^n \frac{\partial H}{\partial p_a^x} \frac{\partial \tilde{c}}{\partial x_a}(X(\alpha)) - \tilde{\beta} \tilde{c}(X(\alpha)) - \sum_{a,b=1}^n \frac{\partial^2 H}{\partial p_a^x \partial p_b^x} \frac{\partial^2 \tilde{g}}{\partial x_a \partial x_b}(X(\alpha)) \\
& + \sum_{a,b=1}^n \frac{\partial^2 H}{\partial p_a^x \partial p_b^x} \frac{\partial \tilde{g}}{\partial x_a}(X(\alpha)) \frac{\partial \ln J}{\partial x_b}(X(\alpha)) \\
& - \sum_{a,b,c=1}^n \frac{\partial \tilde{g}}{\partial x_a}(X(\alpha)) \frac{\partial^3 H}{\partial p_a^x \partial p_b^x \partial p_c^x} \frac{\partial^2 S}{\partial x_b \partial x_c}(X(\alpha)) - \int d\mu_\beta \sum_{a,b=1}^n \frac{\partial^2 \tilde{g}}{\partial x_a \partial x_b}(X(\beta)) \frac{\partial^2 H_1}{\partial p_a^y \partial p_b^y} \\
& - \int d\mu_\beta \sum_{a,b,c=1}^n \frac{\partial \tilde{g}}{\partial x_a}(X(\beta)) \left( \frac{\partial^2 S}{\partial x_a \partial x_b}(X(\alpha)) \frac{\partial^3 H_1}{\partial p_a^y \partial p_b^x \partial p_c^x} + \frac{\partial^2 S}{\partial y_a \partial y_b}(X(\beta)) \frac{\partial^3 H_1}{\partial p_a^y \partial p_b^y \partial p_c^y} \right) \\
& + \int d\mu_\beta \sum_{a,b=1}^n \frac{\partial \tilde{g}}{\partial x_a}(X(\beta)) \left( \frac{\partial \ln J}{\partial x_b}(X(\alpha)) \frac{\partial^2 H_1}{\partial p_a^y \partial p_b^x} + \frac{\partial \ln J}{\partial x_b}(X(\beta)) \frac{\partial^2 H_1}{\partial p_a^y \partial p_b^y} \right) \\
& - i \int d\mu_\beta \tilde{c}(X(\beta)) \sum_{a=1}^n \left( \frac{\partial \ln J}{\partial x_a}(X(\alpha)) \frac{\partial H_1}{\partial p_a^x} + \frac{\partial \ln J}{\partial x_a}(X(\beta)) \frac{\partial H_1}{\partial p_a^y} \right) \\
& + i \int d\mu_\beta \sum_{a=1}^n \frac{\partial \tilde{c}}{\partial x_a}(X(\beta)) \frac{\partial H_1}{\partial p_a^y} \\
& + i \int d\mu_\beta \tilde{c}(X(\beta)) \sum_{a,b=1}^n \left( \frac{\partial^2 H_1}{\partial p_a^x \partial p_b^x} \frac{\partial^2 S}{\partial x_a \partial x_b}(X(\alpha)) + \frac{\partial^2 H_1}{\partial p_a^y \partial p_b^y} \frac{\partial^2 S}{\partial x_a \partial x_b}(X(\beta)) \right) \\
& + \frac{h}{2} \sum_{a,b,c=1}^n \frac{\partial \tilde{c}}{\partial x_a}(X(\alpha)) \frac{\partial^3 H}{\partial p_a^x \partial p_b^x \partial p_c^x} \frac{\partial^2 S}{\partial x_b \partial x_c}(X(\alpha)) \\
& - \frac{h}{2} \sum_{a,b=1}^n \frac{\partial \tilde{c}}{\partial x_a}(X(\alpha)) \frac{\partial^2 H}{\partial p_a^x \partial p_b^x} \frac{\partial \ln J}{\partial x_b}(X(\alpha)) + \frac{h}{2} \sum_{a,b=1}^n \frac{\partial^2 H}{\partial p_a^x \partial p_b^x} \frac{\partial^2 \tilde{c}}{\partial x_a \partial x_b}(X(\alpha)) \\
& + \frac{h}{2} \int d\mu_\beta \sum_{a,b,c=1}^n \frac{\partial \tilde{c}}{\partial x_a}(X(\beta)) \left( \frac{\partial^3 H_1}{\partial p_a^y \partial p_b^x \partial p_c^x} \frac{\partial^2 S}{\partial x_b \partial x_c}(X(\alpha)) + \frac{\partial^3 H_1}{\partial p_a^y \partial p_b^y \partial p_c^y} \frac{\partial^2 S}{\partial x_b \partial x_c}(X(\beta)) \right) \\
& + \frac{h}{2} \int d\mu_\beta \sum_{a,b=1}^n \left[ \frac{\partial^2 H_1}{\partial p_a^y \partial p_b^y} \left( \frac{\partial^2 \tilde{c}}{\partial x_a \partial x_b}(X(\beta)) - \frac{\partial \tilde{c}}{\partial x_a}(X(\beta)) \frac{\partial \ln J}{\partial x_b}(X(\beta)) \right) \right. \\
& \left. - \frac{\partial \tilde{c}}{\partial x_a}(X(\beta)) \frac{\partial \ln J}{\partial x_b}(X(\alpha)) \frac{\partial^2 H_1}{\partial p_a^y \partial p_b^x} \right] = 0. \tag{A.13}
\end{aligned}$$

If  $H_0(x, p_x) = p_x^2/2 + U(x)$  and  $H_1(x, p_x; y, p_y) = V(x, y)$ , then Eqs. (A.10) and (A.13) become much simpler and acquire the form

$$\begin{aligned}
& (i\nabla S \nabla - \tilde{\beta})\tilde{g} + \int V(x, X(\alpha')) \tilde{c}(X(\alpha')) d\mu_{\alpha'} + \frac{h}{2}(-\Delta \tilde{g} + \nabla \ln J \nabla \tilde{g}) = 0, \\
& (i\nabla S \nabla - \tilde{\beta})\tilde{c} - \Delta \tilde{g} + \nabla \ln J \nabla \tilde{g} - \frac{h}{2}(-\Delta \tilde{c} + \nabla \ln J \nabla \tilde{c}) = 0.
\end{aligned} \tag{A.14}$$

From Eqs. (A.14) one can approximately find the functions  $F$  and  $G$ , which are important for constructing approximate wave functions in the  $N$ -particle problem as  $N \rightarrow \infty$  [5].

Let us now relate the obtained results to the solution to variational equation for the Vlasov equation, obtained in the preceding part of this paper [3].

Let  $\hat{\rho}$  be the projection on the function  $\psi$ . Its kernel is  $\tilde{\rho}(x, y) = \psi(x)\psi^*(y)$ , and its symbol is  $\rho(x, p) = \psi(x)\tilde{\psi}^*(p)e^{(i/\hbar)px}$ . The operator  $\hat{\rho}$  satisfies the Wigner equation, which reduces to the Vlasov equation as  $\hbar \rightarrow 0$ . The operator  $\hat{\sigma}$  with the kernel  $F(x)\psi^*(y) + \psi(x)G(y)$  is equal to

$$\hat{\sigma} = \tilde{f}\hat{\rho} + \hat{\rho}\tilde{g} \quad (\text{A.15})$$

and satisfies the variational equation to the Wigner equation, which is reduced to the variational equation for the Vlasov equation (20) obtained in [3]. In Eq. (A.15)  $\tilde{f}$  and  $\tilde{g}$  are the operators of multiplication by the functions  $\tilde{f}$  and  $\tilde{g}$ . We see that in the semiclassical approximation the symbol of  $\sigma$  is  $O(\hbar)$ , since  $\hat{\sigma} = [\hat{\rho}; \tilde{g}] + \hbar \tilde{c}\hat{\rho}$  and

$$\sigma(x, p) \simeq \hbar \left( -i \sum_{a=1}^n \frac{\partial \rho}{\partial p_a}(x, p) \frac{\partial \tilde{g}}{\partial x_a} + \tilde{c}\rho \right).$$

Since  $\rho$  is the  $\delta_\Lambda$ -function in the semiclassical approximation [3], the function  $\sigma$  is actually the sum of the  $\delta_\Lambda$ -function and its derivative. Equations (A.14) are consistent with Eqs. (24) obtained in [3] for the coefficients of  $\delta$  and  $\delta'$ . Thus, the approach suggested in this part allows us to find an asymptotic formula for  $\sigma$  as well.

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